

# Locus and Optimization Problems in Lower and Higher Dimensions

Alasdair McAndrew

Alasdair.McAndrew@vu.edu.au

College of Engineering and Science

Victoria University

PO Box 18828, Melbourne 8001

Victoria, Australia

Wei-Chi Yang

wyang@radford.edu

Department of Mathematics and Statistics

Radford University

Radford, VA 24142

USA

## Abstract

*In this paper we consider a locus problem that originated in a practice examination for admission to Chinese universities. We have made the problem more interesting and challenging by adding an optimization component. We also extend the locus and optimization problems to sphere and hyperspheres. The original problem looks simple and yet can prove challenging to many students. The solutions for locus in 2D are accessible to high school students. The solutions for maximizing the area of rectangular box using Lagrange multipliers is accessible to university students who have learned multi-variable calculus. Finally, optimization using Gröbner bases can be understood by those graduate students who have grasped the concept.*

## 1 Introduction

The following problem is typical of a *Gaokao* practice problem [1, 6]: one which is set on the University entrance examinations held across China. Although the problems may differ between provinces, and between rural and urban settings, they are designed to be challenging.

We consider the following problem which is modified from the original practice problem [1]. Suppose a circle of radius 1 is centred at the origin, and a point  $C = (a, 0)$  with  $0 < a < 1$  is chosen. Let  $E$  be a point on the circle, and  $F$  another point so that the angle  $FCE$  is a right angle. Let  $G$  be chosen so that  $CEGF$  is a rectangle. The problem is to determine the locus of  $G$  as  $E$  moves around the circumference of the circle. We remark that the original problem stated in [1] is when the radius  $r = 6$  and  $a = 4$ . To make the problem more challenging for university students, we follow up by asking the maximum value of the area of the rectangle  $CEGF$ .

The problem is illustrated in Figure 1.

In this paper, we use a Dynamic Geometry Software (DGS) for exploring the problems and gain some intuitions for possible solutions. Next we apply a Computer Algebra System (CAS) to verify that our conjectures are valid.

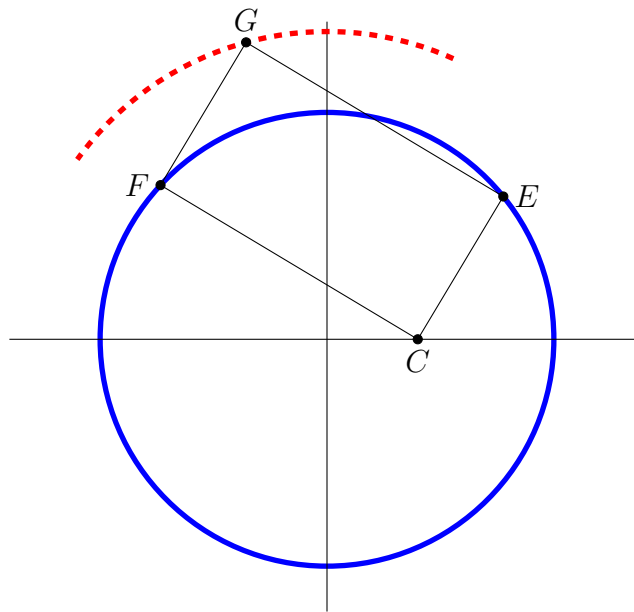


Figure 1: The problem

## 1.1 Explore with a DGS

We use a DGS such as GeoGebra [2] to conjecture what the locus will be and when the area yields a maximum. It is not hard to set up the initial circle and points, and then use the `LOCUS` command to obtain the locus. We might then make the guess that the locus was a circle. We can also determine the area of the rectangle, and indeed plot that area against the angle of  $CE$  with the positive  $x$ -axis.

## 2 Finding the locus analytically

To find the locus of  $G$  analytically, we may start by letting the angle at the base of  $CE$  be  $\theta$ . Let the lengths of  $CE$  and  $CF$  be  $u$  and  $v$  respectively. Then the coordinates of  $E$  and  $F$  are:

$$E = (u \cos \theta + a, u \sin \theta), \quad F = (a - v \sin \theta, v \cos \theta).$$

This can be seen in Figure 2.

Since  $E$  and  $F$  are both on a unit circle, it follows that

$$(a + u \cos \theta)^2 + (u \sin \theta)^2 = 1, \quad (v \sin \theta - a)^2 + (v \cos \theta)^2 = 1$$

and these equations can be rewritten as

$$a^2 + 2u \cos \theta + u^2 - 1 = 0, \quad a^2 - 2v \sin \theta + v^2 - 1 = 0 \tag{1}$$

By elementary symmetry,

$$G = E + F - C = (u \cos \theta + v \sin \theta - a, u \sin \theta + v \cos \theta).$$

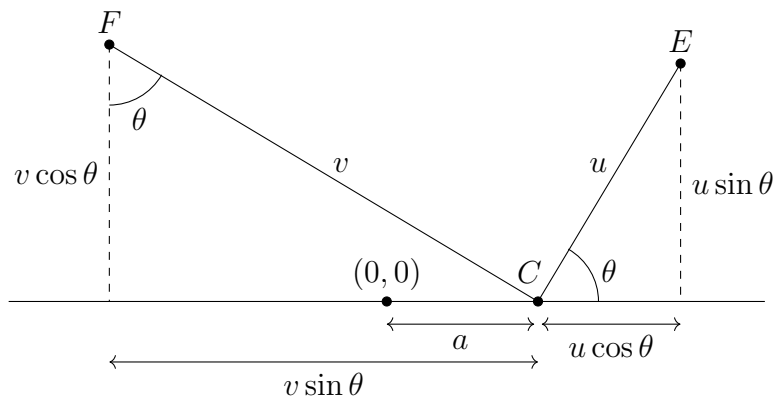


Figure 2: Lengths and angles

It is now not hard to show that the distance of  $G$  to the origin is independent of  $\theta$ . To start:

$$(u \cos \theta + v \sin \theta - a)^2 + (u \sin \theta + v \cos \theta)^2 = u^2 + v^2 + a^2 + 2au \cos \theta - 2av \sin \theta \quad (2)$$

and the right hand side is easily obtained by expansion and repeated use of the identity  $\sin^2 \theta + \cos^2 \theta = 1$ . However, equations (1) can be rewritten as

$$u^2 + 2au \cos \theta = 1 - a^2, \quad v^2 - 2av \sin \theta = 1 - a^2.$$

Substituting these into the right hand side of equation (2) produces  $2 - a^2$  which is the square of the distance of  $G$  from the origin. Thus the locus of  $G$  is a circle centred at the origin and of radius  $\sqrt{2 - a^2}$ .

A neater way involves vectors. Let  $OC$  be  $\mathbf{a}$ ,  $CE$  and  $CF$  be  $\mathbf{u}$  and  $\mathbf{v}$  respectively, and  $OE$  and  $OF$  be  $\mathbf{E}$  and  $\mathbf{F}$  respectively. These are shown in Figure 3.

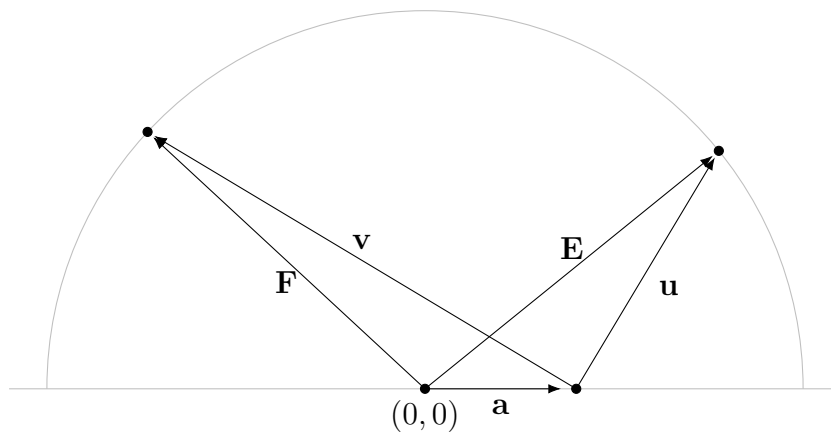


Figure 3: Using vectors

Since by construction  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular, their scalar product is zero:  $\mathbf{u} \cdot \mathbf{v} = 0$ . We also have that

$$\mathbf{E} = \mathbf{a} + \mathbf{u}, \quad \|\mathbf{E}\| = 1, \quad \mathbf{F} = \mathbf{a} + \mathbf{v}, \quad \|\mathbf{F}\| = 1. \quad (3)$$

The external point is obtained as

$$\mathbf{G} = \mathbf{a} + \mathbf{u} + \mathbf{v}.$$

Then

$$\begin{aligned} \|\mathbf{G}\|^2 &= \mathbf{G} \cdot \mathbf{G} = (\mathbf{a} + \mathbf{u} + \mathbf{v}) \cdot (\mathbf{a} + \mathbf{u} + \mathbf{v}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + 2\mathbf{a} \cdot \mathbf{u} + 2\mathbf{a} \cdot \mathbf{v} + 2\mathbf{u} \cdot \mathbf{v} \\ &= \|\mathbf{a}\|^2 + \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{a} \cdot \mathbf{u} + 2\mathbf{a} \cdot \mathbf{v}. \end{aligned} \quad (4)$$

However, from equations (3) we have

$$\|\mathbf{E}\|^2 = \|\mathbf{a} + \mathbf{u}\|^2 = (\mathbf{a} + \mathbf{u}) \cdot (\mathbf{a} + \mathbf{u}) = 1$$

or

$$\mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} = \|\mathbf{a}\|^2 + \|\mathbf{u}\|^2 + 2\mathbf{a} \cdot \mathbf{u} = 1$$

and similarly

$$\|\mathbf{a} + \mathbf{v}\|^2 = \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{a}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{a} \cdot \mathbf{v} = 1.$$

From equation (4) we have

$$\begin{aligned} \|\mathbf{G}\|^2 + \|\mathbf{a}\|^2 &= 2\|\mathbf{a}\|^2 + \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{a} \cdot \mathbf{u} + 2\mathbf{a} \cdot \mathbf{v} \\ &= \|\mathbf{a} + \mathbf{u}\|^2 + \|\mathbf{a} + \mathbf{v}\|^2 \\ &= 2. \end{aligned}$$

Thus

$$\|\mathbf{G}\|^2 = 2 - \|\mathbf{a}\|^2$$

which is equivalent to our previous result.

### 3 Optimizing the area

Again we can use the DGS to help determine the positions and values of the maximum and minimum areas: the area  $A = (CE)(CF)$  can be plotted against  $\theta$  as shown in Figure 4. From this plot we might guess that the maximum area is obtained when  $\theta = 3\pi/4$  and the minimum when  $\theta = -\pi/4$ . For both of these angles  $CE = CF$  and  $G$  lies on the  $x$ -axis.

To find the area of the rectangle, equations (1) can be solved to obtain

$$u = -a \cos \theta \pm \sqrt{1 - a^2 \sin^2 \theta}, \quad v = a \sin \theta \pm \sqrt{1 - a^2 \cos^2 \theta}.$$

Choosing the positive sign in each of these expressions for  $u$  and  $v$  in order to ensure that the values are positive, we have the area of the rectangle being

$$A = (-a \cos \theta + \sqrt{1 - a^2 \sin^2 \theta})(a \sin \theta + \sqrt{1 - a^2 \cos^2 \theta}).$$

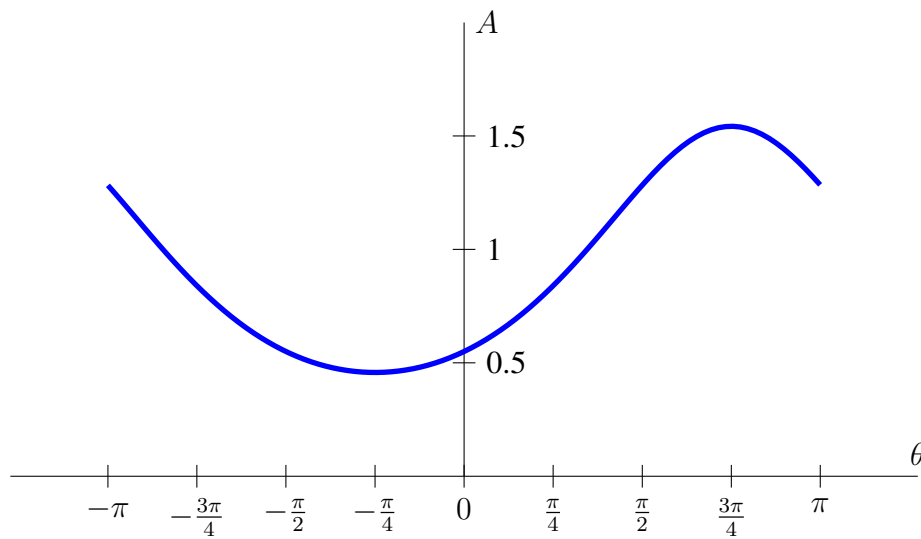


Figure 4: The area of the rectangle

We might guess that the largest rectangle will be obtained when  $\theta = 3\pi/4$  in which case  $u = v$  by symmetry and so the rectangle is a square. At  $\theta = 3\pi/4$  we can find that

$$u = \frac{1}{2}\sqrt{2}(a + \sqrt{2 - a^2})$$

$$v = \frac{1}{2}\sqrt{2}(a + \sqrt{2 - a^2})$$

$$\frac{du}{d\theta} = \frac{\sqrt{2}(a^2 + a\sqrt{2 - a^2})}{2\sqrt{2 - a^2}}$$

$$\frac{dv}{d\theta} = -\frac{\sqrt{2}(a^2 + a\sqrt{2 - a^2})}{2\sqrt{2 - a^2}}$$

Since  $u = v$  and  $u' = -v'$  it follows that  $A' = u'v + uv' = 0$  and hence  $\theta = 3\pi/4$  is a stationary point for  $A$ .

Note that if we write  $R$  for  $\sqrt{2 - a^2}$ , being the radius of the circle which is the locus of  $G$ , then the above four values can be neatly written as

$$u = v = \frac{1}{\sqrt{2}}(a + R), \quad u' = -v' = \frac{a}{\sqrt{2}} \left(1 + \frac{a}{R}\right) = \frac{a}{R\sqrt{2}}(a + R).$$

Since  $A = uv$  then at  $\theta = 3\pi/4$

$$A = \frac{1}{2}(a + R)^2 = \frac{1}{2}(a^2 + 2a\sqrt{2 - a^2} + 2 - a^2) = 1 + a\sqrt{2 - a^2}.$$

The above working is not conceptually difficult, but it is not trivial, and the scope for error in hand calculation is very great. So we might attempt to ameliorate such errors by employing a computer algebra system.

## 4 Using a computer algebra system

Our choice is Sage [4], which is open-source, and has powerful geometric and algebraic capabilities. Indeed, Sage was originally called SAGE, this being the acronym of “Software for Algebraic and Geometric Experimentation”.

All of the steps from the previous sections can be easily implemented, starting by defining the variables to be used and showing that the locus is a circle:

```
sage: var('u,v,x,y,a,theta')
sage: E = {x:a+u*cos(theta),y:u*sin(theta)}
sage: F = {x:a-v*sin(theta),y:v*cos(theta)}
```

Now the values of  $u$  and  $v$  can be computed:

```
sage: usol = solve(E[x]^2+E[y]^2-1,u)
sage: vsol = solve(F[x]^2+F[y]^2-1,v)
```

The external point  $G$  can be defined, and the square of its distance to the origin computed:

```
sage: G = {x:E[x]+F[x]-a,y:E[y]+F[y]}
sage: R = (G[x]^2+G[y]^2).trig_simplify().expand()
sage: R
```

$$2au \cos(\theta) - 2av \sin(\theta) + a^2 + u^2 + v^2$$

Finally the values from the  $u$  and  $v$  solutions above can be substituted:

```
sage: R.subs({u:usol[0].rhs(),v:vsol[0].rhs()}).full_simplify()
```

$$-a^2 + 2$$

### 4.1 Applying Gröbner bases and Lagrange multipliers

An alternate method to determine the locus of  $G$  is to use Gröbner bases. This requires using the ring of polynomials over the rational numbers, in which all the equations defining the points and areas can be written. Suppose the coordinates of  $E$  and  $F$  are  $(x_1, y_1)$  and  $(x_2, y_2)$ . Their gradients to  $C = (a, 0)$  must have a product of  $-1$  so that

$$\frac{y_1}{x_1 - a} \frac{y_2}{x_2 - a} = -1$$

or simply that

$$(x_1 - a)(x_2 - a) + y_1 y_2 = 0.$$

Since  $E$  and  $F$  are on the circle, and  $G$  has coordinates  $(g_1, g_2) = (x_1 + x_2 - a, y_1 + y_2)$  we can find the locus of  $G$  as follows:

```
sage: P.<x1,y1,x2,y2,g1,g2,a> = PolynomialRing(QQ)
sage: onCircle = [x1^2+y1^2-1,x2^2+y2^2-1]
sage: perpendicular = [(x1-a)*(x2-a)+y1*y2]
sage: g_point = [g1-x1-x2+a,g2-y1-y2]
sage: I = P.ideal(onCircle + perpendicular + g_point)
sage: I.elimination_ideal([x1,y1,x2,y2])
```

$$(g_1^2 + g_2^2 + a^2 - 2)\mathbb{Q}[x_1, y_1, x_2, y_2, g_1, g_2, a]$$

This last may be interpreted to be equal to zero, thus

$$g_1^2 + g_2^2 = 2 - a^2$$

which shows that the locus is a circle. Note that here we did not have to make any assumptions about the shape or nature of the locus; we just computed a relationship between the coordinates of  $G$  and the fixed value  $a$ .

To explain the previous approach a little: we have created a polynomial *ideal*; which may be considered as the closure under addition and polynomial multiplication from an initial set—in our case this set consisted of the polynomials which define the relationships between all the coordinates. A *Gröbner basis* may be considered as an alternative spanning set for the ideal consisting of polynomials with decreasing numbers of variables; in a sense a Gröbner basis is analogous to a triangularization of a linear system. In Sage, the `elimination_ideal` method creates a Gröbner basis and returns the polynomial from that basis which is independent of the given variables. An elegant and elementary introduction to Gröbner bases is given by Sturmfels [5].

We can also investigate the maximum size of the rectangle, using Lagrange multipliers. We have to maximize  $A = (CE)(CF)$ , or

$$A^2 = ((x_1 - a)^2 + y_1^2)((x_2 - a)^2 + y_2^2).$$

The constraints are:

$$\begin{aligned}x_1^2 + y_1^2 &= 1 \\x_2^2 + y_2^2 &= 1 \\(x_1 - a)(x_2 - a) + y_1 y_2 &= 0.\end{aligned}$$

Thus we need to solve the simultaneous first derivatives of

$$L = ((x_1 - a)^2 + y_1^2)((x_2 - a)^2 + y_2^2) + \lambda_1(x_1^2 + y_1^2 - 1) + \lambda_2(x_2^2 + y_2^2 - 1) + \lambda_3((x_1 - a)(x_2 - a) + y_1 y_2).$$

Using Sage we can set up the expression and compute its derivatives

```
sage: var('x1,y1,x2,y2,a,lambd1,lambd2,lambd3')
sage: L = ((x1-a)^2+y1^2)*((x2-a)^2+y2^2) + lambd1*(x1^2+y1^2-1)
sage: L += lambd2*(x2^2+y2^2-1) + lambd3*((x1-a)*(x2-a)+y1*y2)
sage: diffs = [diff(L,x1), diff(L,x2), diff(L,y1), diff(L,y2)]
sage: diffs += [diff(L,lambd1), diff(L,lambd2), diff(L,lambd3)]
```

Now to solve the equations, we can try to obtain an expression giving the area in terms only of  $a$ , by using again Gröbner bases:

```
sage: P.<x1,y1,x2,y2,a,lambd1,lambd2,lambd3,K>\
.....: = PolynomialRing(QQ)
sage: I = P.ideal(diffs+[K^2-((x1-a)^2+y1^2)*((x2-a)^2+y2^2)])
sage: E = I.elimination_ideal([x1,y1,x2,y2,lambd1,lambd2,\
.....: lambd3])
sage: Q0 = E.gen(0).factor()[1][0]
sage: var('K')
sage: solve(SR(Q0),K)
```

$$[K = -\sqrt{-a^2 + 2a + 1}, K = \sqrt{-a^2 + 2a + 1}]$$

The second value is clearly positive; it can be written as

$$K = 1 + a\sqrt{2 - a^2}$$

which of course is our original result. For further information about Lagrange multipliers a good reference is Lagrange's own description, which has been translated into English [3].

## 5 Alternative method

Another method of approaching the rectangle size, which will be easier to generalize to higher dimensions as we shall show later, is to consider the distances from each vertex of the rectangle  $CEGF$  to the point  $p = (0, 0)$ . Clearly the distances of  $C, E, F, G$  to  $p$  are  $a, 1, 1, \sqrt{2 - a^2}$  respectively. This is shown on the left in Figure 5, and the rectangle itself, without the circles, is shown on the right.

We introduce the lengths  $s$  and  $t$  as the projections of  $pC$  onto the rectangle sides  $CE$  and  $CF$  respectively. Note that since  $s^2 + t^2 = a^2$ , we have  $\sqrt{1 - s^2} = \sqrt{1 - a^2 + t^2}$  and  $\sqrt{1 - t^2} = \sqrt{1 - a^2 + s^2}$ . Thus the area to be maximized is

$$A = (s + \sqrt{1 - a^2 + s^2})(t + \sqrt{1 - a^2 + t^2})$$

subject to the condition  $s^2 + t^2 = a^2$ . In order to apply the method of Lagrange multipliers, consider the expression

$$B = (s + \sqrt{1 - a^2 + s^2})(t + \sqrt{1 - a^2 + t^2}) + \lambda(s^2 + t^2 - a^2)$$

and its derivatives:

$$\frac{\partial B}{\partial s} = \left(1 + \frac{s}{\sqrt{1 - a^2 + s^2}}\right)(t + \sqrt{1 - a^2 + t^2}) + 2\lambda s \quad (5)$$

$$\frac{\partial B}{\partial t} = (s + \sqrt{1 - a^2 + s^2})\left(1 + \frac{t}{\sqrt{1 - a^2 + t^2}}\right) + 2\lambda t \quad (6)$$

$$\frac{\partial B}{\partial \lambda} = s^2 + t^2 - a^2. \quad (7)$$





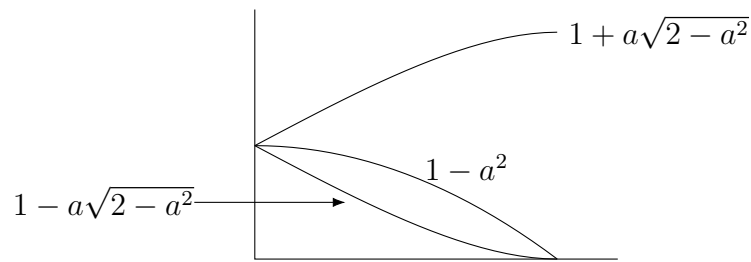


Figure 6: Plots of area functions

## 6 Extending the Problem to Higher Dimensions

The above computations regarding locus can easily be extended to a sphere. Suppose the sphere has radius one, and we choose a point  $C$  in its interior. Let  $E$ ,  $F$  and  $G$  be three points on the surface so that  $CE$ ,  $CF$  and  $CG$  are all pairwise perpendicular. Consider these three lines as the edges of a rectangular cuboid with common vertex  $C$ , and let  $H$  be its opposite vertex. This is illustrated in Figure 7.

Then the locus of  $H$  is a sphere.

In order to see this, we adopt an approach similar to that for the circle, and define the vectors

$$\mathbf{a} = OC, \mathbf{u} = CE, \mathbf{v} = CF, \mathbf{w} = CG, \mathbf{E} = OE, \mathbf{F} = OF, \mathbf{G} = OG, \mathbf{H} = OH.$$

Then

$$\mathbf{a} + \mathbf{u} = \mathbf{E}, \quad \mathbf{a} + \mathbf{v} = \mathbf{F}, \quad \mathbf{a} + \mathbf{w} = \mathbf{G}$$

and

$$\|\mathbf{E}\|^2 = \|\mathbf{F}\|^2 = \|\mathbf{G}\|^2 = 1$$

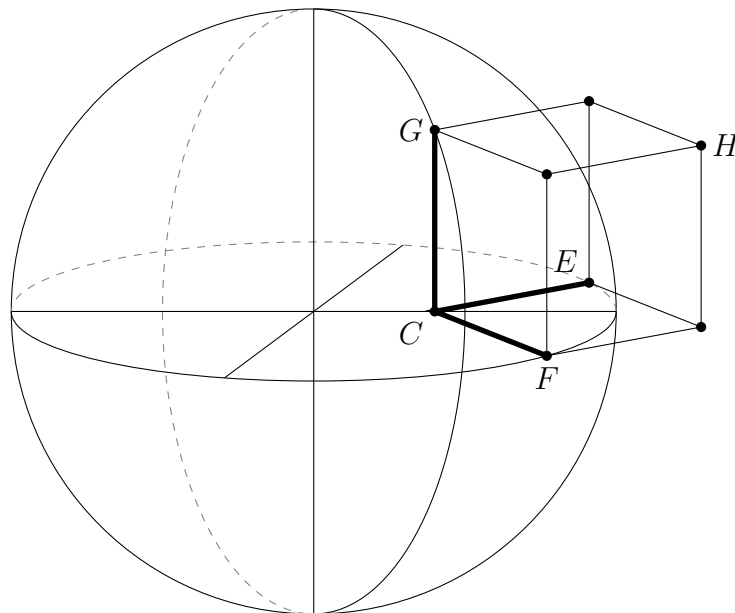


Figure 7: The locus problem on a sphere

and also

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0.$$

Finally,

$$\mathbf{H} = \mathbf{a} + \mathbf{u} + \mathbf{v} + \mathbf{w}.$$

Using the same methods as for the circle,

$$\begin{aligned} \|\mathbf{H}\|^2 + 2\|\mathbf{a}\|^2 &= \|\mathbf{a} + \mathbf{u}\|^2 + \|\mathbf{a} + \mathbf{v}\|^2 + \|\mathbf{a} + \mathbf{w}\|^2 \\ &= \|\mathbf{E}\|^2 + \|\mathbf{F}\|^2 + \|\mathbf{F}\|^2 \\ &= 3 \end{aligned}$$

and so

$$\|\mathbf{H}\|^2 = 3 - 2\|\mathbf{a}\|^2.$$

Again this can be done in Sage. Let the three points on the surface have coordinates  $(x_i, y_i, z_i)$  for  $i = 1, 2, 3$  and let  $C = (a, 0, 0)$ . Perpendicularity can be characterized by the usual scalar product in  $\mathbb{R}^3$ , so that for  $i \neq j$ :

$$\langle x_i - a, y_i, z_i \rangle \cdot \langle x_j - a, y_j, z_j \rangle = 0$$

or that

$$(x_i - a)(x_j - a) + y_i y_j + z_i z_j = 0.$$

We can describe  $\mathbf{H}$  in terms of  $\mathbf{E}$ ,  $\mathbf{F}$ , and  $\mathbf{G}$  by

$$\begin{aligned} H &= \mathbf{a} + \mathbf{u} + \mathbf{v} + \mathbf{w} \\ &= \mathbf{a} + (\mathbf{E} - \mathbf{a}) + (\mathbf{F} - \mathbf{a}) + (\mathbf{G} - \mathbf{a}) \\ &= \mathbf{E} + \mathbf{F} + \mathbf{G} - 2\mathbf{a}. \end{aligned}$$

This means that the coordinates of  $H$  are given as

$$H = (x_1 + x_2 + x_3 - 2a, y_1 + y_2 + y_3, z_1 + z_2 + z_3).$$

Now we can set up all the equations, and simplify them by the computation of a Gröbner basis:

```
sage: P.<x1, y1, z1, x2, y2, z2, x3, y3, z3, g1, g2, g3, a> = PolynomialRing(QQ)
sage: onSphere = [x1^2+y1^2+z1^2-1, x2^2+y2^2+z2^2-1, x3^2+y3^2+z3^2-1]
sage: perpendicular = [(x1-a)*(x2-a)+y1*y2+z1*z2,
(x1-a)*(x3-a)+y1*y3+z1*z3, (x3-a)*(x2-a)+y3*y2+z3*z2, ]
sage: h_point = [g1-x1-x2-x3+2*a, g2-y1-y2-y3, g3-z1-z2-z3]
sage: I = P.ideal(onSphere + perpendicular + h_point)
sage: I.elimination_ideal([x1, y1, z1, x2, y2, z2, x3, y3, z3])
```

$$(g_1^2 + g_2^2 + g_3^2 + 2a^2 - 3)\mathbf{Q}[x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, g_1, g_2, g_3, a]$$

This corresponds with the vector result obtained above: the locus is a sphere with radius  $r^2 = 3 - 2a^2$ . Note that we made no assumptions about the geometry of the locus: the input consisted only of the polynomial relationships between the vertex coordinates; the output was a relationship which

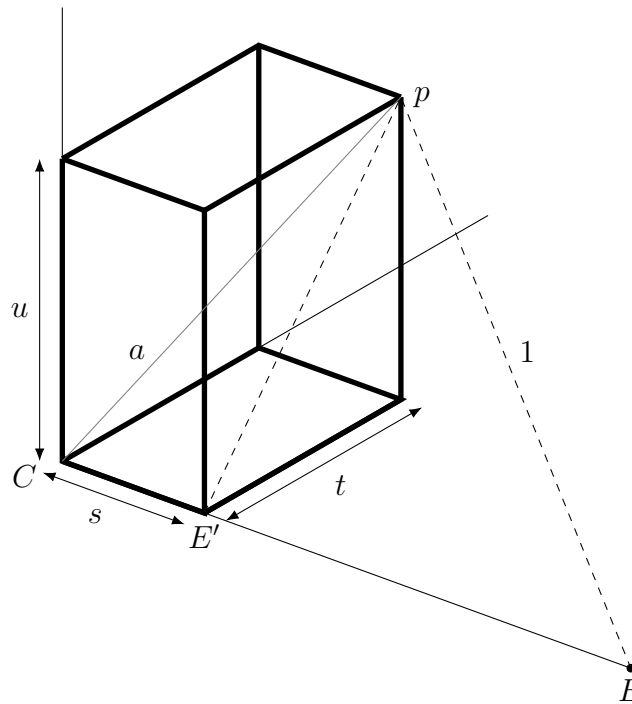


Figure 8: Setting up the three dimensional volume

characterises a sphere. This is similar to our computations for the circle, where again we made no assumptions about the geometry of the locus.

To maximize the volume of the cuboid, suppose that it is aligned with the  $xyz$  axes, and with  $C$  at the origin, and  $p$  at  $(s, t, u)$ , as for the rectangle in Figure 5. The vertex at  $C$  is shown in Figure 8.

From the diagram, we have  $pE' = \sqrt{t^2 + u^2}$  and so

$$E'E = \sqrt{1 - t^2 - u^2} = \sqrt{1 - a^2 + s^2}$$

which means

$$CE = s + \sqrt{1 - a^2 + s^2}.$$

Thus the volume is given by

$$V = (s + \sqrt{1 - a^2 + s^2})(t + \sqrt{1 - a^2 + t^2})(u + \sqrt{1 - a^2 + u^2})$$

and this is to be maximized subject to  $s^2 + t^2 + u^2 = a^2$ . By exactly the same argument with Lagrange multipliers as above, we show that optimal values of  $V$  will be obtained when  $s^2 = t^2 = u^2 = a^2/3$ , and so each of  $s, t, u = \pm a/\sqrt{3}$ . The maximum value will be obtained when  $s = t = u = a/\sqrt{3}$  to obtain edge lengths  $CE, CF$  and  $CG$  all equal to

$$a/\sqrt{3} + \sqrt{1 - a^2 + a^2/3} = \frac{1}{\sqrt{3}}(a + \sqrt{3 - 2a^2})$$

and a volume of

$$\left(\frac{1}{\sqrt{3}}(a + \sqrt{3 - 2a^2})\right)^3 = \frac{\sqrt{3}}{9}(9a - 5a^3 + (a^2 + 3)\sqrt{3 - 2a^2}).$$

Now it is easy to consider  $n$ -dimensions. We suppose  $\mathbf{a}$  is a vector with  $\|\mathbf{a}\| < 1$ , and there are  $n$  vectors  $\mathbf{u}_i$  all of which are pairwise orthogonal, and all of which satisfy  $\|\mathbf{a} + \mathbf{u}_i\| = 1$ . We are to consider the locus of  $\mathbf{g} = \mathbf{a} + \mathbf{u}_1 + \cdots + \mathbf{u}_n$ .

We have

$$\begin{aligned}\|\mathbf{g}\|^2 &= \mathbf{g} \cdot \mathbf{g} \\ &= (\mathbf{a} + \mathbf{u}_1 + \cdots + \mathbf{u}_n) \cdot (\mathbf{a} + \mathbf{u}_1 + \cdots + \mathbf{u}_n) \\ &= \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{u}_1 + \cdots + 2\mathbf{a} \cdot \mathbf{u}_n + \mathbf{u}_1 \cdot \mathbf{u}_1 + \cdots + \mathbf{u}_n \cdot \mathbf{u}_n\end{aligned}$$

and all the other products  $\mathbf{u}_i \cdot \mathbf{u}_j = 0$  by orthogonality. Adding  $(n - 1)\mathbf{a} \cdot \mathbf{a}$  to both sides produces

$$\begin{aligned}\|\mathbf{g}\|^2 + (n - 1)\mathbf{a} \cdot \mathbf{a} &= \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{u}_1 + \mathbf{u}_1 \cdot \mathbf{u}_1 + \\ &\quad \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{u}_2 + \mathbf{u}_2 \cdot \mathbf{u}_2 \\ &\quad + \cdots + \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{u}_n + \mathbf{u}_n \cdot \mathbf{u}_n \\ &= (\mathbf{a} + \mathbf{u}_1) \cdot (\mathbf{a} + \mathbf{u}_1) + (\mathbf{a} + \mathbf{u}_2) \cdot (\mathbf{a} + \mathbf{u}_2) \\ &\quad + \cdots + (\mathbf{a} + \mathbf{u}_n) \cdot (\mathbf{a} + \mathbf{u}_n) \\ &= \|\mathbf{a} + \mathbf{u}_1\|^2 + \|\mathbf{a} + \mathbf{u}_2\|^2 + \cdots + \|\mathbf{a} + \mathbf{u}_n\|^2 \\ &= n\end{aligned}$$

from which we obtain

$$\|\mathbf{g}\|^2 = n - (n - 1)\|\mathbf{a}\|^2.$$

Writing  $\|\mathbf{a}\| = a$  gives us the locus of the external point being an  $n$ -sphere with radius  $r^2 = n - (n - 1)a^2$ .

To maximize the hyper-volume, we note that it is given by the expression

$$H = \prod_{i=1}^n (x_i + \sqrt{1 - a^2 + x_i^2})$$

subject to

$$x_1^2 + x_2^2 + \cdots + x_n^2 = a^2.$$

From the reasoning above, the maximum volume is obtained when all the  $x_i$  values are equal to  $a/\sqrt{n}$ . Thus the maximum volume for  $n$  dimensions will be

$$\left( \frac{1}{\sqrt{n}} (a + \sqrt{n - (n - 1)a^2}) \right)^n.$$

Expressions for  $n = 4, 5, 6$  are:

$$\begin{aligned}V_4 &= \frac{1}{2} \left( 2 - a^4 - (a^3 - 2a)\sqrt{4 - 3a^2} \right) \\ V_5 &= \frac{\sqrt{5}}{125} \left( 41a^5 - 150a^3 + 125a + (25 + 10a^2 - 19a^4)\sqrt{5 - 4a^4} \right) \\ V_6 &= \frac{1}{27} \left( 22a^6 - 45a^4 + 27 + (7a^5 - 30a^3 + 27a)\sqrt{6 - 5a^2} \right)\end{aligned}$$

## 7 Discussions and future work

A nice problem is to explore, using a DGS, what happens to the locus for the circle case if the angle  $FCE$  is changed to something other than 90 degrees. In fact, we can prove that if the angle  $FCE$  is 60 degrees, the locus will not be a circle. The explorations of other angles is a fruitful line of enquiry which the authors will discuss in a future paper. Another possibility for exploration is to replace the circle by an ellipse and consider the following questions:

1. Given an ellipse with a chosen internal point, and with a fixed angle  $FCE$  as in Figure 1, what is the locus of  $G$ ?
2. What is the maximum value of the area of the parallelogram  $CEGF$ ?

Note that because the ellipse is not as symmetric as the circle, we could also choose a different point  $C$  within the ellipse; one that is not on either of the major or minor axes for example.

Extending our sphere scenario to an ellipsoid produces the scenario: we are given a fixed ellipsoid with the major axes,  $OA \geq OB \geq OC$  and an internal point  $C$ . For example we could choose  $C = (a, 0, 0)$  with  $a \leq OC$ . Let  $E = (x_1, y_1, z_1)$ ,  $F = (x_2, y_2, z_2)$ ,  $G = (x_3, y_3, z_3)$  be three points on the ellipsoid. We further suppose that each pair of vectors from  $CE$ ,  $CF$  and  $CG$  have a fixed angle  $\beta$  between them. Considering the parallelepiped formed by  $CE$ ,  $CF$  and  $CG$ , we may ask the following questions:

1. If  $H$  is a symmetric point of  $G$  for the parallelepiped, what is the locus of  $H$ ?
2. Find the largest possible value of the volume of the parallelepiped.

As for the two-dimensional problem above, we could choose other places for  $C$ .

Initial exploration indicates that these problems are difficult to solve in complete generality, which leads to the further problem of considering what ellipsoid parameters, internal point positions, and common angles lead to curves or surfaces which can be easily described.

## 8 Conclusion

Finding a curve defined by the locus of a moving point is popular and often asked in *Gaokao* in China. Typically students are allocated no more than 10 minutes to solve one problem. Under such circumstances, it is not hard to imagine that many students may lose interest and may even decide to give up to solving these types of problems. It is clear that technological tools can provide us with crucial intuition before we attempt more rigorous analytical solutions. Here we have gained geometric intuitions while using GeoGebra. In the meantime, we use the computer algebra system, Sage, for verifying that our analytical solutions are consistent with our initial intuitions. In this paper, we started out with a simple locus problem but we added an optimization component and investigated several scenarios including ones in 2D, 3D and beyond. The complexity level of the problems we posed vary from the simple to the difficult: some of our solutions are accessible to students from high school; others require more advanced mathematics such as university or even graduate levels. Evolving technological tools definitely have made mathematics fun and accessible on one hand, but

they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging.

We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important task for many educators and researchers.

## References

- [1] Shi Gi Gin Bang. “Strategies for High School Mathematics Complete Review”. In: ed. by Cui Jun Guan Zhiming. Century Gold. Yanbian University Press, 2015, p. 647.
- [2] M. Hohenwarter et al. *Geogebra, Dynamic Mathematics for Learning and Teaching, Version 5.0*. IGI. 2015. URL: <http://www.geogebra.org/>.
- [3] Joseph Louis Lagrange. *Analytical Mechanics*. Vol. 191. trans. Auguste Boissonnade and Victor N. Vagliente. Springer Science & Business Media, 2013.
- [4] William A. Stein et al. *Sage Mathematics Software (Version 6.7.0)*. The Sage Development Team. 2015. URL: <http://www.sagemath.org>.
- [5] Bernd Sturmfels. “What is a Gröbner basis?” In: *Notices of the AMS* 52.10 (Nov. 2005), pp. 2–3.
- [6] Wei-Chi Yang, Vladimir Shelomovskii, and Sylvester Thompson. “Extreme Lengths for Chords of an Ellipse”. In: *The Electronic Journal of Mathematics and Technology (eJMT)* 9.2 (2015), pp. 180–191. URL: <https://php.radford.edu/~ejmt/>.